Independently generated languages

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1 Compositionality versus Independence

How do humans understand the meaning of a complex sentence they have never heard before? The answer is that there is an algorithm that allows to compute the meaning of the complex expression from its parts. While this much seems uncontroversial, semanticists have actually argued that natural languages possess a stronger property, that of compositionality. A language is said to be compositional if the meaning of a complex expression is a function of the meaning of its parts given the mode of composition; thus, a language is compositional if the algorithm computing the meaning can do so without knowing the expressions that carry these meanings. It is this latter property that has been made into a litmus test for formal semantic theories. A theory that provides a compositional account of meaning is preferred over one that does not. But how much of a constraint is compositionality on a language? In other words, what empirical significance does it have to say that a language is compositional? Do noncompositional languages at all exist?

While the introduction of the subject is often credited to Montague, it is perhaps the work that has been done in the wake of Montague that had put compositionality firmly on the agenda of formal semantic theorising, see Janssen (1997) for an account by one of the protagonists. The survey books Barker and Jacobson (2007) and Hinzen et al. (2012) document the persistent interest in this notion. From a mathematical point of view, the question is how much of an empirical content this notion has. Janssen has actually shown that any language is compositional, provided no constraints on syntactic operations are being made (see Janssen (1997)). Though his notion of language is slightly nonstandard, the result holds also for languages in the Saussurean sense, ie relations between expressions and meanings.

However, as Kracht (2011) has pointed out, the dual property, namely that the *form* of an expression is independent of the meaning of its parts, is actually a well known hypothesis of generative grammar: It is called the *autonomy of syntax*. It is curious that to our knowlede no definition of this notion with the same explicitness has ever been given in print. In retrospect it seems that a grammar that satisfies both of them simultaneously is really what linguistics have been after, and not compositionality alone. We call this property *independence*. It has a rather simple mathematical formulation, so the investigation may also be of interest in combinatorial theory.

Moreover, it turns out that the property of independence is actually rather tricky. It is still unclear whether there exists a countable language that is not independently generated. Though we believe that such a language exists, we have not been able to find one. The results here exhibit some positive results (specifying languages that are independently generated) and reduces the complexity of the original problem somewhat.

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2 Autonomy and Compositionality

A *language* is an arbitrary subset L of $E \times M$, where E and M are given sets of expressions and meanings, respectively. An *independent grammar* for L is a finite set $F \subseteq L$ and a finite set P of pairs of functions (f_i, g_i), $i < m$, such that for $i < m$ there is an n_i (the arity of the functions) such that $f_i : E^{n_i} \hookrightarrow E$ and $g_i : M^{n_i} \hookrightarrow M$ are both partial functions and L is exactly the set that can be generated from F using P. The action of such a pair is defined as usual: $(f, g)((e_0, m_0), \cdots, (e_{n_i-1}, m_{n_i-1}))$ is defined if and only if both $f(e_0, \cdots, e_{n_i-1})$ and $g(m_0, \cdots, e_{n_i-1})$ are defined and in that case

$$(f, g)((e_0, m_0), \cdots, (e_{n_i-1}, m_{n_i-1})) := (f(e_0, \cdots, e_{n_i-1}), g(m_0, \cdots, m_{n_i-1}))$$

There is an obvious mathematical generalisation. Let $R \subseteq \omega^d$ be a relation. Say that R is *independently generated* if there is a finite set of partial functions f_i^k, $k < d$, of arity n_i (only dependending on i) such that the product functions $(f_i^0, f_i^1, \cdots, f_i^{d-1})$ (of arity n_i) generate R from a finite subset. The limiting case of $d = 1$ is trivial. Any countable subset of ω can be generated; it suffices to pick one constant and a single unary function. Thus, the case $d = 2$ is the first really interesting case. Notice also that if there is a relation of arity d that is not independently generated, then there is an example in any higher arity.

Notation. If f is a function and S a set, put $f[S] := \{f(x) : x \in S\}$. Given $L \subseteq \omega^2$ we write L_i for the "column" $\{j : (i, j) \in L\}$. It follows that $L = \bigcup_{i \in \omega} \{i\} \times L_i$. Dually we write ${}^jL := \{i \in \omega : (i, j) \in L\}$.

The notion of independence is restrictive. Let $L \subseteq E \times M$ be countably infinite. Then there exists a finite subset F and a finite number of partial functions on $E \times M$ (rather than independent functions on E and M) generating L. Indeed, one constant plus a single unary function is enough. Simply enumerate $L = \{(e_i, m_i) : i \in \omega\}$, put $F := \{(e_0, m_0)\}$ and let $f : E \times M \to E \times M$ be defined by

$$f((e,m)) := \begin{cases} (e_{i+1}, m_{i+1}) & \text{if } (e,m) = (e_i, m_i) \\ (e,m) & \text{else} \end{cases}$$

By assumption, i is unique in the first case. Then it is easily seen that $(e_k, m_k) = f^k((e_0, m_0))$, so we generate exactly L.

However, the question was whether it is possible to define the functions in such a way that the actions on E and on M are independent of each other. In the linguistic literature, two weaker notions have been discussed. The most important one is compositionality. The first result will be that all countable languages have a compositional grammar.

Since the set of generating functions is finite, L is at most countably infinite. Thus we can restrict E and M to some countably infinite subset; without loss of generality we can assume them to be the set of natural numbers $\omega = \{0, 1, 2, 3, \cdots\}$. (Formally, there is nothing that distinguishes members of E from members of M.) Thus, from now on $E = M = \omega$. Let $p_1 : \omega^2 \to \omega : (i, j) \mapsto i$ and $p_2 : \omega^2 \to \omega : (i, j) \mapsto j$.

Definition 1 *Let f be a partial n-ary function on ω^2. We say that f is* ***independent in the first component*** *if for all pairs $(i_0, j_0), \cdots, (i_{n-1}, j_{n-1})$ and all pairs $(i_0, k_0), \cdots, (i_{n-1}, k_{n-1})$: $f((i_0, j_0), \cdots, (i_{n-1}, j_{n-1}))$ is defined if and only if $f((i_0, k_0), \cdots, (i_{n-1}, k_{n-1}))$ is defined, and if any of them is defined, then the first projection of the values are identical, ie*

$$p_1(f((i_0, j_0), \cdots, (i_{n-1}, j_{n-1}))) = p_1(f((i_0, k_0), \cdots, (i_{n-1}, k_{n-1}))).$$

Dually the notion of ***independence in the second component*** *is defined.*

Theorem 2 *Let $L \subseteq \omega^2$. Then there is a finite set of functions generating L from a finite subset where all functions are independent in the first component. Likewise, the is a finite set of functions generating L from a finite subset where all functions are independent in the second component.*

Proof. Obviously, we need to prove only the first claim. The second follows analogously. (Or instead, apply the method to $L^{\smile} := \{(j, i) : (i, j) \in L\}$ and then "switch" the solution.) Now, consider first the language

$$M := \{(i, j) : L_i \neq \varnothing, j \text{ is minimal in } L_i\}.$$

Clearly, $M \subseteq L$.

So M has the form $M = \{(i, n_i) : i \in H\}$ for some $H \subseteq \omega$. Let α be the least member of H, ie the least number such that $L_\alpha \neq \varnothing$. Introduce a constant for (α, n_α). Next, let $k(i, j)$ be defined as follows. In case $j = n_i$ and there is a $q > i$ such that $L_q \neq \varnothing$, let $k(i, j) := (p, n_p)$, where p is the smallest number $> i$ such that L_p is nonempty. If no such number exists, or if $j \neq n_i$, put $k(i, j) := (i, j)$. Now put

$$d((i, j)) := (p, k(i, j))$$

This defines our first function. (A) d is independent in the first component since p can be established from i alone. (B) L is closed under d. For given $(i, j) \in L$, if $d((i, j)) = (i, j)$ then obviously $d((i, j)) \in L$. If $d((i, j)) \neq (i, j)$, then $(i, j) = (i, n_i)$ and $d((i, j))$ therefore has the form (p, n_p), where by definition $(p, n_p) \in M$. (C) M is the closure of $\{(\alpha, n_\alpha)\}$ under d. We prove by induction on i that all $(i, n_i) \in M$ can be generated. For $i = \alpha$ this is the case by assumption. Let i be given with $(i, n_i) \in M$. Then let $p < i$ be the largest number such that $(p, n_p) \in M$. By inductive hypothesis, (p, n_p) is generated from $\{(\alpha, n_\alpha)\}$. But $(i, n_i) = d((p, n_p))$, so it is also generated from $\{(\alpha, n_\alpha)\}$.

Next, define a function ν as follows. Given i and j, $\nu(i, j) := j$ if either $j \notin L_i$ or j is the largest member of L_i. Otherwise, $\nu(i, j)$ yields the least j' such that $j' \in L_i$ and $j' > j$. Now put

$$u((i, j)) := (i, \nu(i, j))$$

(A) u is evidently independent in the first component. (B) L is closed under u. For if $(i, j) \in L$ and $u((i, j)) = (i, j)$ then $u((i, j)) \in L$. Otherwise, $u((i, j)) = (i, \nu(i, j)) = (i, j')$, where among other $j' \in L_i$. So, $u((i, j)) \in L_i \subseteq L$. (C) L is generated from M using u. This is proved by induction on j. Choose $(i, j) \in L$. If j is minimal in L_i then $j = n_i$ and the claim trivially follows. Otherwise, choose j' to be maximal such that $(i, j') \in L$ and $j' < j$. By inductive hypothesis, (i, j') is generated from M using u. But $u((i, j')) = (i, j)$, and so (i, j) is likewise generated from M using u. □

Notice that we have been able to define total functions. Consider a system F of generating (partial) functions on $E \times M$. This system is called **compositional** if each member $f \in F$ is independent in the second component; it is called **autonomous** if each member $f \in F$ is independent in its first component. We can rephrase the previous theorem as follows. A language is **compositional** (**autonomous**) if it has a finite compositional (autonomous) set of generating functions.

Corollary 3 *Let L be a countable language.*

- *L is autonomous.*
- *L is compositional.*

□

Here is a surprising consequence.

Corollary 4 *Suppose that L is either many-to-one (=unambiguous) or L is one-to-many. Then L is independently generated.*

Proof. Consider the second case, ie assume that L is one-to-many (the other case being dual). Let $f : M \to E$ be such that $f(j)$ is the unique i such that $(i, j) \in L$. Define the grammar as in the proof of Theorem 2. Now put

$$d((i, j)) := (p, k(f(j), j))$$

as well as

$$u((i, j)) := (i, \nu(f(j), j))$$

where k and ν are defined as before. By assumption, $i = f(j)$, so that this defines the same function. The so-defined functions do not depend on the first component any more, and so are independent. □

Corollary 5 *Let $L \subseteq E \times M$ a language such that for some $A \subseteq E$, $B \subseteq M$, $L \cap A \times B$ is a many-to-one or one-to-many relation on $A \times B$ containing an infinite partial bijection. Then L is independently generated.*

Proof. We generate $L \cap A \times B$ by means of independent partial functions defined on $A \times B$, as shown in Corollary 4. L contains a partial infinite bijection $\{(a_i, b_i) : i < \omega\}$. Let $L - A \times B = \{(e_i, m_i) : i < \omega\}$. Now add a new unary partial function $f : A \times B \to (E - A) \times (M - B)$ with

$$f((x, y)) = \begin{cases} (e_i, m_i) & \text{if } (x, y) = (a_i, b_i) \\ \text{undefined} & \text{else} \end{cases}$$

By assumption, i is uniquely determined by x alone *and* by y alone, so f is actually independent. □

The notion of independence for languages is not the conjunction of autonomy and compositionality (if it were, all languages would be independent, by Corollary 3); indeed, it is much stronger than that. For it says that the language has an independent grammar, that is, a grammar where every function is independent in both components. This is what we are going to study now.

3 Basic Results

In using partial functions, here is a trick that will be used on several occasions. Denote by $[F]_P$ the closure of F under P. Let A be the disjoint union of B and C. Let P be a set of partial functions on B, and Q a set of partial function on C. Take $B_0 \subseteq B$ and $C_0 \subseteq C$. Then $[B_0 \cup C_0]_{P \cup Q} = [B_0]_P \cup [C_0]_Q$. To see that notice that functions from P are undefined on every tuple containing elements from C, and functions from Q are undefined on every tuple containing an element from B. Therefore, functions from P cannot act on outputs of functions from Q, and vice versa.

Lemma 6 *Let $L \subseteq \omega^2$, and $\omega = E' \cup E''$, with E' and E'' disjoint, and let $M \subseteq \omega$. Now put $L' := L \cap E' \times M$, $L'' := L \cap E'' \times M$. Then if both L' and L'' are independently generated, so is $L \cap \omega \times M$.*

Indeed, simply take the (disjoint) union of the constants and functions. The following two claims are obvious.

Lemma 7 *If L is independently generated, so is $L^{\smile} := \{(j, i) : (i, j) \in L\}$.*

Lemma 8 *Let $\pi, \rho : \omega \to \omega$ be injections. Let $(\pi, \rho)[L] := \{(\pi(e), \rho(m)) : (e, m) \in L\}$. Then $(\pi, \rho)[L]$ is independently generated iff L is independently generated.*

There is a special corollary of this theorem that is worth stating separately. Consider the case where $L_i = \varnothing$ for certain i. Denote by $U := \{i : L_i \neq \varnothing\}$. If U is infinite there is a bijection $\nu : U \to \omega$. Consider the language $L^\bullet := (\nu, id_M)[L]$. We have $(L^\bullet)_i = L_{\nu(i)} \neq \varnothing$ for all $i \in \omega$.

Corollary 9 *$L^\bullet$ is independently generated iff L is.*

If U is finite, L is independently generated anyway, by the next theorem.

Lemma 10 *Let n be a finite number.*

1. *Every finite language is independently generated.*
2. *$n \times n$ is independently generated.*
3. *$\omega \times \omega$ is independently generated.*
4. *$\omega \times n$, $n \times \omega$ are independently generated.*
5. *Every cofinite subset of $\omega \times \omega$ is independently generated.*

Proof. The first claim is easy. Just introduce a constant for every element of L. The second claim follows immediately. To show the third claim introduce a constant for $(0, 0)$, and two unary functions: one sending (i, j) to $(i + 1, j)$, and one sending (i, j) to $(i, j + 1)$. The fourth claim is proved thus. For each $j < n$ take a constant for $(0, j)$. Finally, add a single unary function sending (i, j) to $(i + 1, j)$. For the last claim, let $L = \omega \times \omega - \{(i_k, j_k) : k < n\}$. Put $E_0 := \{i_k : k < n\}$, $E_1 := \omega - E_0$; $M_0 := \{j_k : j < n\}$, $M_1 := \omega - M_0$. Now $L \cap E_0 \times M_0$ is finite; $L \cap E_0 \times M_1 = E_0 \times M_1$, $L \cap E_1 \times M_0 = E_1 \times M_0$, and $L \cap E_1 \times M_1 = E_1 \times M_1$. The first is independently generated since it is finite. The others are independently generated because they are a simple product of at most countable sets. Now use Lemma 6. □

Say that L is *essentially bounded* if $L \subseteq n \times \omega$ or $L \subseteq \omega \times n$.

Lemma 11 *Every essentially bounded language is independently generated.*

Proof. From Lemma 10 by repeated application of Lemma 6. □

Next we are going to reduce of the problem even further. Let $H \subseteq \omega$ such that for every $i \in H$ there is a $j \notin H$ and $L_j = L_i$. Then put

$$L^{-H} := \{(i, j) : (i, j) \in L, i \notin H\}$$

Lemma 12 *If L^{-H} is independently generated then L is independently generated.*

Proof. Suppose that L^{-H} is independently generated. For $j \notin H$ put

$$B_j := \{k : k \in H, L_k = L_j\}$$

Now let $h : \omega \to \omega$ be defined as follows. If $j \notin H$ and $B_j = \varnothing$ then $h(j) := j$. If $j \notin H$ and $B_j \neq \varnothing$, then let $h(j) := \min B_j$. Else, if $j \in H$ then $j \in B_i$ for some i.

If $j = \max B_i$, put $h(j) := j$, otherwise let $h(j)$ be the least $j' \in B_i$ such that $j' > j$. Finally, let f be defined by

$$f((i, j)) := (h(i), j)$$

(A) f is independent. (B) L is closed under f. Consider $(i, j) \in L$. If $f((i, j)) = (i, j)$ then $f((i, j)) \in L$. Otherwise, $f((i, j)) = (h(i), j)$, where by definition $L_{h(i)} = L_i$. Thus, $(h(i), j) \in L_{h(i)}$ and so $(h(i), j) \in L$. (C) L is generated from L^{-H} using f. If not, let i be minimal such that for some j, $(i, j) \in L$ but it is not generated from L^{-H} using f. Then $i \in H$. Let i' the largest number such that $i' < i$ and $i' \in H$ if it exists, else let $i' \notin H$ such that $i = \min B_{i'}$. By definition, $i = h(i')$. It is easily seen that (i', j) is generated from L^{-H} using f; the same is now true for (i, j). □

Thus we can restrict our search for nonindependent languages to those subsets of ω^2 where all columns are different and all rows are different.

4 Main Theorems

We are going to investigate three conditions under which languages are independently generated. The second and third conditions both generalise the first, in slightly different directions. Examples will show that the generalisations are proper.

Definition 13 *Let $L \subseteq \omega \times \omega$. Say that L is n-**discriminable** if there is a family $\{A_i : i \in \omega\}$ of sets such that:*

1. *for each i: $0 < |A_i| \leq n$;*
2. *for each i, j: if $i \neq j$ then $A_i \neq A_j$;*
3. *for every i: $A_i \subseteq L_i$; and*
4. *for every i, j $A_j \subseteq L_i$ if and only if $j = i$.*

*In that case, we call the family $\{A_i : i \in \omega\}$ an n-**discriminating family** for L. (Notice that 4. implies 3.)*

Notice that by definition, $A_i \not\subseteq A_j$ for $i \neq j$. For if $A_i \subseteq A_j$ we have $A_i \subseteq A_j \subseteq L_j$, from which by definition $i = j$.

Theorem 14 *Let L be n-discriminable. Then L is independently generated.*

Proof. Let $\{A_i : i \in \omega\}$ be a discriminating family for L. Let $\overline{A}_i$ be a sequence of length n that enumerates A_i, possibly repeating an element to reach length n. (Eg if $n = 4$ and $A_2 = \{3, 6, 7\}$ then $\overline{A}_2 = \langle 3, 6, 7, 7\rangle$ is a possible choice.) For each member of $\{(0, i) : i \in A_0\}$ we introduce a constant. Now we define the following n-ary functions f^k, $k < n$. Let h be the kth member of $\overline{A}_{i+1}$.

$$(1) \qquad f^k((i, j_0), (i, j_1), \cdots, (i, j_{n-1})) := \begin{cases} (i+1, h) & \text{if } \langle j_0, j_1, \cdots, j_{n-1}\rangle = \overline{A}_i \\ \text{undefined} & \text{else} \end{cases}$$

Clearly this function is independent: on the first component it gives $i+1$ if all arguments are identical to i and is undefined otherwise. On the second component it gives h if the arguments are exactly given as in $\overline{A}_i$, and is undefined else. The partiality seems to be essential here.

Now define a single $n+1$-ary function g with the following action. For each i we assume $L_i - A_i$ to be enumerated as $\{k^i_j : j < \kappa_i\}$ where $\kappa_i < \omega + 1$ (so κ_i can be finite or $= \omega$).

$$(2) \qquad g((i, j_0), (i, j_1), \cdots, (i, j_n)) := \begin{cases} (i, k^i_0) & \text{if } \langle j_0, j_1, \cdots, j_{n-1}\rangle = \overline{A}_i \\ & \quad \text{and } j_n = j_{n-1}, \kappa_i \neq 0 \\ (i, k^i_{p+1}) & \text{if } \langle j_0, j_1, \cdots, j_{n-1}\rangle = \overline{A}_i \\ & \quad \text{and } j_n = k^i_p, p+1 < \kappa_i \\ \text{undefined} & \text{else} \end{cases}$$

So defined g is independent. On the first coordinate the output is i if all inputs equal i; and is undefined elsewhere. On the second coordinate it yields the next element in the enumeration if there is one (and repeats the element if it is the last in the enumeration), provided the first n arguments equal $\overline{A}_i$; and is undefined elsewhere.

It now remains to be shown that this set of functions generates exactly L. There are two parts: (i) the functions generate all of L; (ii) L is closed under the functions.

To prove (i), we shall first show that all $\{i\} \times A_i$ are generated using the f^k and the constants. Recall that all members of $\{0\} \times A_0$ are values of some constant. Now by induction assume that $\{i\} \times A_i$ is generated. Thus all pairs (i, j_p) exist, $p < n$, where $j_p \in A_i$. Then, using the functions f^k, we can generate $(i+1, h)$, where h is the kth member of $\overline{A}_{i+1}$. Since all elements of A_{i+1} appear at least once in $\overline{A}_{i+1}$, all of $\{i+1\} \times A_{i+1}$ is thus generated. Next we show that for every i, $\{i\} \times L_i$ is generated from $\{i\} \times A_i$ using the function g. To that end, recall that $L_i - A_i$ is enumerated as k^i_0, k^i_1 and so on for indices in κ_i. If $\kappa_i = 0$, nothing needs to be done. If $\kappa_i > 0$, we get (i, k^i_0) as the value of $g((i, j_0), (i, j_1), \cdots, (i, j_{n-1}), (i, j_{n-1}))$, and (i, k^i_{p+1}) as the value of $g((i, j_0), (i, j_1), \cdots, (i, j_{n-1}), (i, k^i_p))$. By induction, all values are generated.

Finally, we need to show that L is closed under the functions. Consider

$$f^k((i_0, j_0), (i_1, j_1), \cdots, (i_{n-1}, j_{n-1}))$$

This is defined only if $i := i_0 = i_1 = \cdots = i_{n-1}$ and $\langle j_0, j_1, \cdots, j_{n-1}\rangle = \overline{A}_q$ for some q. We have $q = i$, since $\{i\} \times A_p \subseteq L$ only if $p = i$ by definition of n-discrimination. So, the function is defined only on $f^k((i, j_0), (i, j_1), \cdots, (i, j_{n-1}))$, where $\langle j_0, \cdots, j_{n-1}\rangle = A_i$, and yields the value $(i+1, j'_k)$, where j'_k is the kth member of $\overline{A}_{i+1}$. By definition, this is in L. Next, consider

$$g((i_0, j_0), \cdots, (i_n, j_n))$$

This is defined only if $i := i_0 = i_1 = \cdots = i_n$. Additionally, like in the case of f^k, the sequence $\langle j_0, j_1, \cdots, j_{n-1}\rangle$ must equal $\overline{A}_i$. Hence we have to look at

$$g((i, j_0), (i, j_1), \cdots, (i, j_n))$$

Several cases arise. (a) $j_n = j_{n-1}$. In that case we get (i, k^i_0), provided that $\kappa_i > 0$. In that case, $L_i - A_i$ is nonempty, and contains k^i_0 by definition. (b) $j_n = k^i_p$, where k^i_p is a member of $L_i - A_i$. In fact, it is the pth member of the enumeration. If $p + 1 = \kappa_i$, then $L_i - A_i$ is exhausted, and g is undefined. If not, we get (i, k^i_{p+1}), which is in $L_i - A_i$ by definition. (c) The function is undefined on all other inputs. In all cases, we get values in L. The proof is complete. □

Corollary 15 *Suppose that there exists an n such that for all $i \in \omega$ $|L_i| \leq n$. Then L is independently generated.*

Proof. By Lemma 12 we can reduce this to the case where for $i \neq j$ $L_i \neq L_j$. Define $I(j) := \{i : |L_i| = j\}$ and $L^j := \cup_{i \in I(j)} L_i$. By Lemma 6, we need to show only that each of the L^j is independently generated. To this end, it is enough to show that $\{L^j_i : i \in I(j)\}$ is a j-discriminating family. This is easy to verify. □

As an application, consider the language $L = \{(i, i), (i, i^2) : i \in \omega\}$. Here we can simply take $A_i := L_i$. Indeed, this is a 2-discriminating family. For $|A_i| \leq 2$, the sets are nonempty, pairwise distinct ($\{i, i^2\} = \{j, j^2\}$ iff $i = j$), and, finally, if $\{i, i^2\} \subseteq L_j$ then $j = i$; for if $\{i, i^2\} = \{j, j^2\}$ then either the sets contain both two members, and then since $i < i^2$, $j < j^2$ we have $i = j$; or they contain one member and then have the form $\{i\} = \{j\}$, from which again $i = j$. So, by the previous result the language is independently generated.

A more complex example, to which this result cannot be applied, though, is $\{(i, i^k) : i, k \in \omega\}$. It is a consequence of the next theorem that this language is independently generated.

Definition 16 *Call a language **weakly** n**-discriminable** if there is a family $\{A_i : i \in \omega\}$ of sets such that*

1. *for every i, $0 < |A_i| \leq n$;*
2. *for every i, j: if $i \neq j$ then $A_i \neq A_j$; and*
3. *for every i, $A_i \subseteq L_i$; and*
4. *for every i, j: if $A_j \subseteq L_i$ then $L_i \subseteq L_j$. (This is trivially true if $i = j$.)*

In particular, if $A_i \subseteq A_j$ then we must have $L_j \subseteq L_i$. Clearly, all n-discriminable languages are also weakly n-discriminable, but the converse does not hold, as the example just given shows. For if L is n-discriminable, we must have $L_i \not\subseteq L_j$ for all $i \neq j$. (For if $i \neq j$ and $L_i \subseteq L_j$, then since $A_i \subseteq L_i$ we also have $A_i \subseteq L_j$, which is excluded.) But the language $\{(i, i^k) : i, k \in \omega\}$ fails this: we have $L_2 \subseteq L_4$. On the other hand, the family defined by $A_i := \{i, i^2\}$ is a weakly discriminating family. For if $A_i \subseteq L_j$ then $i = j^p$ for some p, whence $L_i = \{i^k : k \in \omega\} \subseteq \{j^p : p \in \omega\}$. The next theorem establishes that this language is independently generated.

Theorem 17 *Let L be weakly n-discriminable. Then L is independently generated.*

Proof. Let $M := \{i :$ *for no $j < i$: $A_j = A_i$*$\}$. Furthermore, let $B(i) = \{j : A_i = A_j\}$. Thus, M consists of all minimal members of the sets $B(i)$. Now let m and n be unary partial functions with the following action. $m(j)$ is undefined if j is maximal in M, and otherwise it is $m(j) := \min\{k : k \in M, j < k\}$. $n(j)$ is undefined if j is maximal in $B(j)$, and $n(j) := \min\{k : k \in B(j), k > j\}$ otherwise.

We need three sets of functions in addition to constants for the members of $\{0\}\times A_0$. The first contains the functions f^k, $k < n$. Define the sequences $\overline{A}_i$ as above, with the exception that we require $\overline{A}_i = \overline{A}_j$ if $B(i) = B(j)$ (that is, if $A_i = A_j$).

$$f^k((i, j_0), (i, j_1), \cdots, (i, j_{n-1})) := \begin{cases} (m(i), h) & \text{if } i \in M, \langle j_0, j_1, \cdots, j_{n-1}\rangle = \overline{A}_i \\ \text{undefined} & \text{else} \end{cases} \tag{3}$$

The second set consists of the h^k, $k < n$.

$$h^k((i, j_0), (i, j_1), \cdots, (i, j_{n-1})) := \begin{cases} (n(i), i_k) & \langle j_0, j_1, \cdots, j_{n-1}\rangle = \overline{A}_i \\ \text{undefined} & \text{else} \end{cases} \tag{4}$$

Finally, we define the function g as above:

$$g((i, j_0), (i, j_1), \cdots, (i, j_n)) := \begin{cases} (i, k^i_0) & \text{if } \langle j_0, j_1, \cdots, j_{n-1}\rangle = \overline{A}_i \\ & \quad \text{and } j_n = j_{n-1}, \kappa_i \neq 0 \\ (i, k^i_{p+1}) & \text{if } \langle j_0, j_1, \cdots, j_{n-1}\rangle = \overline{A}_i \\ & \quad \text{and } j_n = k^i_p, p + 1 < \kappa_i \\ \text{undefined} & \text{else} \end{cases} \tag{5}$$

These functions are independent.

Again, we need to show that (i) L is generated from the functions, and (ii) L is closed under these functions. As for (i), we note that by definition, we can generate all A_i where $i \in M$ from $\{0\} \times A_0$. Next, we can generate the $\{j\} \times A_j$ for all $j \in B(j)$ just by applying the h^k, since we have generated its minimal members. Third, by using g we generate the columns L_i.

Now we show that L is also closed under the functions. Consider

$$f^k((i_0, j_0), (i_1, j_1), \cdots, (i_{n-1}, j_{n-1}))$$

This is defined only if $i := i_0 = i_1 = \cdots = i_{n-1} \in M$ and $\langle j_0, j_1, \cdots, j_{n-1}\rangle = \overline{A}_q$ for some q. By definition of M, for two numbers $p, q \in M$, $A_p \neq A_q$, and so $q = i$. The value $(m(i), h)$ is in $A_{m(i)}$ by definition of f^k. Next consider

$$h^k((i_0, j_0), (i_1, j_1), \cdots, (i_{n-1}, j_{n-1}))$$

This is defined only if $i := i_0 = i_1 = \cdots = i_{n-1}$, and $\langle j_0, j_1, \cdots, j_{n-1}\rangle = \overline{A}_q$ for some q. The value is $(n(i), j_k)$; while j_k is again in $\overline{A}_q$, the new index is $n(i)$. However, by choice of the function n, $A_{n(i)} = A_i$, so we get a value from $A_{n(q)}$. Thus, these functions are only defined on $\bigcup_q \{q\} \times A_q$ and yield values in that set.

Finally, we need to show that L is closed under g. Consider

$$g((i_0, j_0), \cdots, (i_n, j_n))$$

This is defined only if $i := i_0 = i_1 = \cdots = i_n$ and $\langle j_0, \cdots, j_{n-1}\rangle = \overline{A}_q$ for some q. Now, suppose that $A_q = \{j_0, \cdots, j_{n-1}\} \subseteq L_i$. Then by assumption on weak discriminability, $L_q \subseteq L_i$. Hence, two cases arise. (i) $q = i$. Then by definition of g, the value is in L_i. (ii) $p \neq i$. Then, since the value is in L_q, and $L_q \subseteq L_i$, it is also in L_i. □

Definition 18 *Call L **boundedly discriminable** if there are numbers n and n', an infinite set $M \subseteq \omega$ and a family $\{A_i : i \in M\}$ of sets such that the following holds:*

1. *for each $i \in M$, $|A_i| \le n$;*
2. *for each $i \in M$: $A_i \subseteq L_i$;*
3. *for each $i \in M$, the set $B(i) := \{j : A_i \subseteq L_j\}$ has at most n' elements; and*
4. *for each $i, j \in M$, $i \neq j$, $B(i) \cap B(j) = \varnothing$.*

Every n-discriminable language is boundedly discriminable; just take $M := \omega$. The sets $B(i)$ each have only one member in this case, so $n' := 1$.

Actually, it follows that for each $i, j \in M$, $i \neq j$, $A_i \neq A_j$. For if $i, j \in M$ and $i \neq j$, the last clause implies $j \notin B(i)$, that is, $A_j \nsubseteq L_j$, so that $A_j \neq A_i$, since $A_i \subseteq L_i$.

Notice that allowing M to be finite would not gain anything, as then the set of indices would be finite, bounded by some multiple of $|M|$. So the only remaining interesting case is where M is infinite. Moreover, we could assume $n = n'$ to simplify the definition.

Theorem 19 *Suppose that L is boundedly discriminable. Then L is independently generated.*

Proof. Without loss of generality we may assume that all the $i \in M$ are minimal in $B(i)$; in particular, the least element of $B(0)$ is $0 \in M$. If $i \notin M$ let $j < i$ be the least element of $B(i)$. Then $j \in M$ and we put $A_j := A_i$.

First we introduce constants for $\{0\} \times A_0$. Next we introduce functions f^k and h^k, $k < n$, as in the previous proof. Finally, for $k < n'$, let z^k be an $n+1$-ary function, defined similar to g above. Let $P(i,k)$ be the statement: *i is the kth number in $B(i)$*. As before, order the elements of $L_i - A_i$ for each $i \in \omega$, and align the elements of A_i in a sequence k^i_j of length n.

$$(6) \qquad z^k((i, j_0), (i, j_1), \cdots, (i, j_n)) := \begin{cases} (i, k^i_0) & \text{if } \langle j_0, j_1, \cdots, j_{n-1}\rangle = \overline{A}_i \\ & \quad P(i,k) \text{ and } j_n = j_{n-1}, \kappa_i \neq 0 \\ (i, k^i_{p+1}) & \text{if } \langle j_0, j_1, \cdots, j_{n-1}\rangle = \overline{A}_i \\ & \quad P(i,k) \text{ and } j_n = k^i_p, p+1 < \kappa_i \\ \text{undefined} & \text{else} \end{cases}$$

It remains that to show that (i) the functions generate L, (ii) L is closed under these functions. (i) is reasonably clear. We generate A_i, $i \in M$, using the functions f^k, and then all the A_i using the g^k as in the previous proof. Finally, the z^k allow to generate all of L_i for $P(i,k)$. Since for each number i there is a $k < n'$ such that $P(i,k)$, we generate L_i from A_i using z^k essentially as we used g. Now on to (ii). Closure under f^k. Consider

$$f^k((i_0, j_0), (i_1, j_1), \cdots, (i_{n-1}, j_{n-1}))$$

This is defined only if $i := i_0 = i_1 = \cdots = i_{n-1} \in M$ and $\langle j_0, \cdots, j_{n-1}\rangle \in \overline{A}_i$; and in that case it yields the kth member of $\overline{A}_j$, j the next member of M. Closure under h^k. Pretty much as in the previous proof. Closure under z^k. Consider

$$z^k((i_0, j_0), (i_1, j_1), \cdots, (i_n, j_n))$$

If this is defined, $i := i_0 = i_1 = \cdots = i_n$, i is the kth member of $B(i)$, and $\overline{A}_q = \langle j_0, \cdots, j_{n-1}\rangle$ for some q. If z^k is defined, we know from q alone the identity of i. Thus, L_i is known in the second component. Now if $A_q \subseteq L_i$, then $q \in B(i)$ and so $A_q = A_i$. Now by assumption either $j_n = j_{n-1}$, and we get the the least member of $L_i - A_i$ according to the enumeration (if $\kappa_i > 0$). Or else we get the next member according to the enumeration. □

5 Progressive Functions

The method has so far been to enumerate L by going through the L_i in increasing order. The interest in this method stems from using grammars to generate languages. We think of a grammar as producing complex expressions from less complex expressions. In that sense, a formation step produces a strictly more complex expression. Consider now an ordering of the E of expressions in increasing complexity. Extend this to a linear order, and number the expressions with natural numbers. We expect now that the meaning of expression j is produced from some of the expressions $0, 1 \cdots, j-1$. This way of generating expressions is called *progressive*.

Definition 20 *Let f be a partial n-ary function on ω. A **point of progressivity** is a vector $\vec{x}$ such that $f(\vec{x}) > \max_{1\leq i\leq n} x_i$ (henceforth simply written $\max\vec{x}$). A **point of stagnation** is a vector $\vec{x}$ such that $f(\vec{x}) = \max\vec{x}$. A **point of regression** is a vector $\vec{x}$ such that $f(\vec{x}) < \max\vec{x}$. f is called **strictly progressive** if it has no points of stagnation or regression. f is **weakly progressive** if it has no points of regression. Finally, a set of functions is strictly or weakly progressive if all its members are.*

We extend this now to functions on ω^2 as follows. If f is an independent function on ω^2 then it has the form $(f_1(\vec{x}), f_2(\vec{y}))$. We say that f is (strictly, weakly) progressive if f_1 is.

The functions in the previous proofs have generally been weakly progressive. The following theorem shows why we cannot strengthen this to strongly progressive functions.

Theorem 21 *There is a $L \subseteq \omega^2$ which cannot be generated by a finite strictly progressive set of independent partial functions.*

Proof. Suppose that F is a finite set of strictly progressive independent partial functions. Let γ be the cardinality of F, and ζ the maximal arity of these functions. We may wlog assume that $\gamma = \zeta$. Then by progressivity, a member from L_i is obtained by applying a function to the members of $\bigcup_{j<i} L_j$. If their number is bounded by k_i, then there are at most ζk_i^ζ elements. Thus, choose the following sequence of numbers.

$$\begin{aligned}\rho_0 &:= 1\\ \rho_{i+1} &:= (i+1)\rho_i^{i+1}\end{aligned}$$

This sequence is strictly increasing. Moreover, for each choice of γ and ζ there is i such that

$$\rho_{i+1} > \gamma\left(\sum_{j\leq i}\rho_i\right)^{\zeta}$$

To see this, note that $\sum_{j\leq i}\rho_i \leq i\rho_i \leq \rho_i^2$, since $\rho_i > i$ (except for $i = 0, 1, 2$). Then

$$\rho_{2\zeta+1} = (2\zeta+1)\rho_{2\zeta}^{2\zeta+1} > (2\zeta+1)\left(\sum_{j\leq i}\rho_{2\zeta}\right)^{\zeta}$$

Now define

$$L := \{(i, j) : j < \rho_i\}$$

Then $|L_i| = \rho_i$ for all i. It follows that for $i = 2\zeta + 1$ there are not enough functions to generate the elements of $L_{2\zeta+1}$ for the elements with lower index. □

6 Conclusion

We have shown that all countable languages are compositional and autonomous. Moreover, some results have been obtained concerning languages that are independently generated. However, it is open whether all countable languages are independently generated. It is also unclear whether or not allowing partial functions rather than total functions makes a difference.

The conjecture is that there exist nonindependent countable languages. However, no example has been found.

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