INDEFINITE DESCRIPTIONS IN TYPED LAMBDA CALCULUS

ZOLTÁN MOLNÁR

ABSTRACT. The epsilon calculus seems to be an appropriate environment for modelling the meaning theory of definite and indefinite descriptions in a natural language. A philosopher of language may ask that whether Russell's meaning theory on descriptions is applicable in this language or not. Or more precisely, in what circumstances has a sentence (containing an epsilon-expression) a contextual meaning, and what is its logically equivalent quantified reformulation. The question was answered for first order languages earlier, but the conditions was full of technical complications and the construction applied difficult semantics. In this paper, the question is answered in a typed lambda calculus, in an easier way and by a simpler semantics.

1. INTRODUCTION

1.1. **Hilbert's epsilon, descriptions and FOL.** The first-order language (FOL) extended by the Hilbertian variable binding operator ε is possibly a good choice as an environment modelling a couple of formal linguistic and language philosophical phenomena concerning descriptions.[1] The term

$$(\varepsilon x)\varphi \tag{1}$$

where φ is a FOL formula and x is a variable, has the following intuitive meaning:

„an F, if there is any F at all" (2)

where predicate F is the intended meaning (or the natural language translation, if one likes it) of the formula φ. The intuition above comes straightforward from the *first epsilon (or transfinite) axiom* introduced by David Hilbert, which is the following formula scheme

$$(\exists x)\varphi(x) \to \varphi((\varepsilon x)\varphi(x))$$

This intended meaning of the epsilon term shows that $(\varepsilon x)\varphi$ can be called *conditional indefinite description,* since "an F" alone is an indefinite description, but adding the conditional clause "there is any F at all" it becomes a different linguistic entity with, perhaps, a different meaning. Obviously, I do not have to mention that the meaning of the phrase "an F" is itself a problematic one. Therefore, the problem of the semantic difference between "an F" and "an F, if there is any F at all" is also a tricky one. In the paper I am committed to the standpoint that these phrases have the same meaning.

In order to show an application of Hilbert's symbol let me reconstruct formally and analyse the sentence

The man drinking a martini is interesting-looking. (3)

in FOL extended by ε (this extended language is denoted by FOL+ε).[2] Since, FOL+ε does not contain definite descriptions, the phrase "the man drinking a martini" can be seen as a special case of the use of

Key words and phrases. lambda calculus, epsilon symbol.

[1] See [Slat07], [Knee, p. 100].

[2] The original sentence can be found in [Donn].

epsilon. A possible solution is to add a uniqueness clause to the formula 'man drinking a martini(x)' (the formula in FOL+ε expressing the natural language predicate "... is a man and drinks a martini".)[3]

$$(\varepsilon x)(\text{man drinking a martini}(x)\ \&\ (\forall y)(\text{man drinking a martini}(y)\ \equiv\ (x = y))$$

Let us denote the term above by

$$(\varepsilon_{\mathrm{D}} x)\text{man drinking a martini}(x) \tag{4}$$

Then the sentence (3) is formulated as follows

$$\text{interesting-looking}((\varepsilon_{\mathrm{D}} x)(\text{man drinking a martini}(x))) \tag{5}$$

Let me remark again, the claim that the phrase "the man drinking a martini" can be expressed by $(\varepsilon_{\mathrm{D}} x)\text{man drinking a martini}(x)$ is not an obvious one, however a possibly good enough working hypothesis. Without a man holding martini in his hand, the meaning of "the man drinking a martini" is as vague as the meaning of the phrase "the man drinking a martini, if anybody at all".

Accepting the hypothesis above, one can formally analyse the act referring to the interesting-looking person in question, even if he holds a glass of water in his hand. In this case, the reference of the term (4) is a person – not drinking a martini – who seems to be interesting.

The problem of sentence (5) reminds one of Russell's Theory of Descriptions (RTD). In Russell's *On Denoting* or in [Whit] it is proposed that descriptions must not to be considered as proper names, but incomplete parts of quatified sentences.

> Thus we must either provide a denotation in cases in which it is at first sight absent, or we must abandon the view that the denotation is what is concerned in propositions which contain denoting phrases. [Russ, p. 484]
>
> According to the view I advocate, a denoting phrase is essentially *part* of a sentence, and does not, as like most single words, have any significance on its own account. [Russ, p. 488]

According to RTD descriptions, as denoting phases, are not interchangeable by identical ones, since they are meaningless in separation, but have only contextual meanings. Russell in the *On Denoting* gives a FOL reformulation for the sentences of the form (3),[4] but in the general case, when the natural language sentence contains more than one descriptions or a lot of logical operators the FOL reformulation can be carried out several ways. One must mind the scope of logical operators and descriptions. Hence, in the general case RTD is rather a FOL reformulation program, in the spirit of the treatment of the simple case described in [Russ]. At this point, a bit naive question arises.

(6) Is the formal sentence (5) equivalent to a plain, quantified one?

If it is in general, then RTD, or better say its quantificational program, is applicable to FOL+ε, in the sense that the denoting phrases, containing the sentence, can be considered as incomplete parts of a quantified reformulation.[5] If it is not in general, then for FOL+ε Donnellan's proposal holds (i.e. sometimes descriptions have referential meaning too).[6] The answer seems to be the latter. The term $(\varepsilon x)\varphi$ is a referring one (its semantic value is always defined) and its semantic is unproblematic, even if there is no φ, at least the reference of $(\varepsilon x)\varphi$ is not a φ. Nevertheless, be careful, RTD is a proposed strategy for

[3] See [Slat09] p. 417.

[4] [Russ].

[5] The crucial point in the tradition of RTD is not that what the FOL reformulation is, but that is there any such reformulation. For instance, as Zvolenszky questioned: "Initially, at issue was the meaning of a specific, rather narrow class of expressions, incomplete definite descriptions: are they devices of reference or of quantification?" [Zvol, p. 1].

[6] See [Donn].

the problem 'how to deal with descriptions' and not even a (mathematical) thesis. $(\varepsilon x)\varphi$, in FOL+ε, is a proper name (in the sense meant by Russell), hence the question must be rewritten in a weak form. But what will be this weakened question?

It is well-known that if the truth value of the sentence $\psi((\varepsilon x)\varphi)$ in any model does not depend on the semantic value of $(\varepsilon x)\varphi$, then there is a FOL reformulation of $\psi((\varepsilon x)\varphi)$.[7] Hence, if $\psi((\varepsilon x)\varphi)$ is an epsilon-invariant formula (its semantic value is independent from the value of the containing epsilon-term) then the term $(\varepsilon x)\varphi$ can be eliminated from $\psi((\varepsilon x)\varphi)$, up to logical equivalence. The problem is that, this plain FOL reformulation is not an explicit or transparent one. The proof of the theorem applies Craig's Interpolation Theorem, which is a pure existence theorem not giving the needed explicit formula. Hence, in the light of the above considerations, the relevant question is the following.

> Is there an explicit, transparent, well-explainable FOL reformulation of $\psi((\varepsilon x)\varphi)$, provided that $\psi((\varepsilon x)\varphi)$ is epsilon-invariant (in some model)?

For FOL+ε, the question has been positively answered in [Moln], however under a lot of technical conditions. When one changes FOL to lambda calculus the picture becomes much more clear. The point is that, in FOL the substitution $\psi[x/(\varepsilon x)\varphi]$ is only a meta language operation, but in the lambda calculus it is encoded into the object language via the application MN, where N is supposed to be an epsilon-term of the form $(\varepsilon x)P$.

1.2. **Hilbert's epsilon and the lambda operator.** In Section 2, it will be given a syntax and semantics for the epsilon symbol in the context of typed lambda calculus (TL). The syntactic notions will be the well-known ones, but in the definitions different way will be followed, based on the tree-technique in the spirit of Curry–Howard Isomorphism.[8] Since, by the Curry–Howard Correspondence, TL is closely related to the proof theory of the natural deduction system of the propositional logic, a possibility is realised to define the TL syntax notions in the same style as proofs. The form of the definitions will fit to this doctrine and a tree-based method will be applied.

In Section 3, it will be seen that in TL the result can be reached much more faster than in FOL. No need to refer to the so called intensional and substitutional epsilon semantics.[9] The strategy will be the following. The typed lambda language extended by Hilbert's epsilon ($\mathcal{L}_\lambda^{\forall\varepsilon}$) will be considered as a formal model of the fragment of the natural language containing descriptions. Then, if it is possible, the epsilon expressions will be eliminated and the sentences containing them will be mapped, in an explicit way, to the epsilon-free quantified reduct $\mathcal{L}_\lambda^{\forall}$ of $\mathcal{L}_\lambda^{\forall\varepsilon}$. The plain lambda language reformulation will keep the logical truth in the model. Giving Montague semantics to the extended language and to the plain epsilon-free language as models (the $(\mathfrak{M}, f)$-s and the $\mathfrak{M}$-s below, respectively) the construction will be unproblematic.

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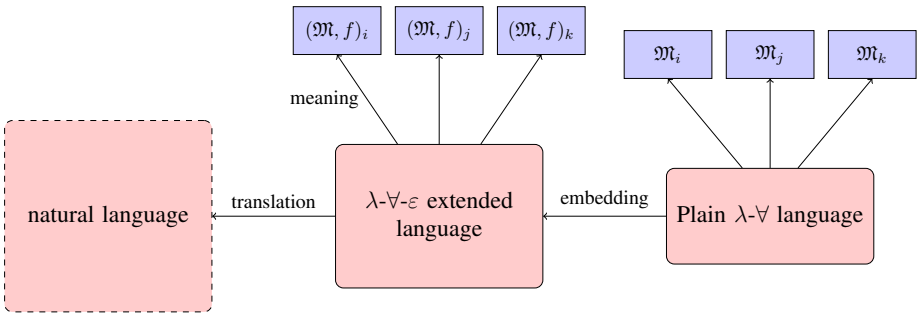
[7] One can call it *Caicedo's Theorem* or the *Blass–Gurevich Theorem*. Its proof first presented in [Caic], but [Blas] claims before the theorem (Prop. 3.2.) that the proposition is a folklore and "it is mentioned in [Caic] without a reference".

[8] It is not easy to refer to a single book or paper, but the book [Simm] (with the programmatic subtitle 'Taking the Curry-Howard Correspondence Seriously') surely uses the tree technique, that I follow.

[9] Note that, Ahrendt and Giese introduced several types of epsilon semantics. See [Ahre, Def. 4,5]. In [Moln] the substitutional semantics was applied. Now, in TL, the extensional semantics (see [Moln, p.821.] or [Monk, Def. 29.23, p. 481]) will be enough.



Chain of fragments. The natural language, the λ-$\forall$-ε expressible fragment and the λ-$\forall$ expressible fragment.

In Section 4, it will pointed out that the result is not less effective than the RTD proposed by Russell.

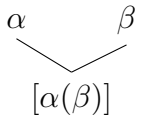
2. Syntax and Semantics of Typed Lambda System with Epsilon

2.1. **Syntax.** For the building of the syntax a tree-based method was chosen (parsing or construction trees), which is much more transparent than the old fashioned character sequence technique. Mind that, here the trees grow upward. Sorry for the inconvenience, if you are a mathematician expert of the lambda calculus. Usually, upward growing trees are used by linguists in Combinatory Categorial Grammar. And, what is the main motivation, such trees are used in the natural deduction style proof theory.

The definitions below are basically combinations of the well-known ones from [Troe] Sec. 1. and from the online lecture notes Sorensen, M. H. B., Urzyczyn, P., *Curry–Howard Isomorphism* (1998)[10].
The so called *typeability relation* ($\vdash$) is a pure syntactic relation which joins the expressions of the lambda calculus to types with respect to a fixed set of typed variables called *context.* Of course, the relation $\vdash$ plays a fundamental role in the Curry–Howard Isomorphism, which links the lambda expressions to proof trees of the natural deduction system of the implicational logic.

Definition 1. The *language of types* is the tuple $\mathcal{L}_{\mathrm{Typ}} = \langle \iota, o, (,), [,] \rangle$. The set of its strings $\mathrm{Srt}(\mathcal{L}_{\mathrm{Typ}})$ contains the finite sequences of characters from $\{\iota, o, (,), [,]\}$. A *construction tree* Π of the string $\gamma \in \mathrm{Srt}(\mathcal{L}_{\mathrm{Typ}})$ is a finite, labeled, ordered tree such that the labels of Π are from $\mathrm{Srt}(\mathcal{L}_{\mathrm{Typ}})$ and

(1) the labels of the leaves of Π come from the set $\{\iota, o\}$,
(2) the branch nodes of Π (these are not leaves) and their labels are of the form



(3) the root of Π is γ.

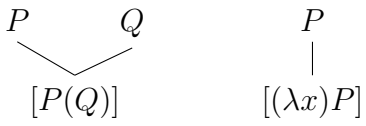
If there is a tree Π such that Π is a construction tree of $\alpha \in \mathrm{Srt}(\mathcal{L}_{\mathrm{Typ}})$, then α is said to be a *type (expression)* in $\mathcal{L}_{\mathrm{Typ}}$. The set of all types in $\mathcal{L}_{\mathrm{Typ}}$ is denoted by $\mathrm{Exp}(\mathcal{L}_{\mathrm{Typ}})$. (Cf. [Troe, p. 9] Def. 1.2.1, [Troe, p. 7] Def. 1.1.7.)

Note that the construction tree of a type is unique. The construction tree of the type α is denoted by $\mathrm{Tree}(\alpha)$. The referring to brackets $[,]$ is avoided when a type α is well-known and its construction tree can be completely reconstruct without them.
Intuitively, ι is the type of individuals and o is the type of sentences. The compound type $o(\iota)$ is, for example, the grammatical type of the single-variable predicates.

[10] `http://disi.unitn.it/~bernardi/RSISE11/Papers/curry-howard.pdf`.

Definition 2. A *lambda language* is a tuple $\mathcal{L}_\lambda = \langle V, C, (,), \lambda, [,] \rangle$, where V is an infinite and C is non-empty set and V is disjoint to C. $\mathrm{Srt}(\mathcal{L}_\lambda)$ contains the finite sequences from $V \cup C \cup \{\lambda, (,), [,]\}$. A *construction tree* Π of the $M \in \mathrm{Srt}(\mathcal{L}_\lambda)$ is a finite, labeled, ordered tree such that the labels of Π are from $\mathrm{Srt}(\mathcal{L}_\lambda)$ and

(1) the labels of the leaves of Π come from the set $V \cup C$,
(2) the branch nodes of Π and their labels are of the form



(3) the root of Π is M.

If there is a tree Π such that Π is a construction tree of $M \in \mathrm{Srt}(\mathcal{L}_\lambda)$, then M is said to be an *expression* in $\mathcal{L}_\lambda$. The set of all expression in $\mathcal{L}_\lambda$ is denoted by $\mathrm{Exp}(\mathcal{L}_\lambda)$.
The elements of V are called the variables of $\mathcal{L}_\lambda$ and V is denoted by $\mathrm{Var}(\mathcal{L}_\lambda)$ and the members of C are the constants of $\mathcal{L}_\lambda$ and C is denoted by $\mathrm{Const}(\mathcal{L}_\lambda)$. (Cf. [Troe, p. 9] Def. 1.2.2, [Troe, p. 7] Def. 1.1.7.)

Note that, the construction tree of an expression is unique. The construction tree of the expression M is denoted by $\mathrm{Tree}(M)$. The *height* of $\mathrm{Tree}(M)$ is defined by the well-known manner and is denoted by $|\mathrm{Tree}(M)|$.
Referring to brackets $[,]$ is avoided, when an expression M is known and its construction tree can be completely reconstruct without them.

Definition 3. Let $\langle V, C, (,), \lambda, [,] \rangle$ be a lambda language. The tuple $\mathcal{L}_\lambda = \langle V, C, (,), \lambda, [,], Z \rangle$ is a *typed lambda language,* if $Z : C \to \mathrm{Exp}(\mathcal{L}_{\mathrm{Typ}})$. The function Z is denoted by $\mathrm{CnstTp}(\mathcal{L}_\lambda)$.

Definition 4. Let $\mathcal{L}_\lambda$ be a lambda language and let $\Xi \subseteq \mathrm{Var}(\mathcal{L}_\lambda)$ be a non-empty finite set. A function $f : \Xi \to \mathrm{Exp}(\mathcal{L}_{\mathrm{Typ}})$ is called a *context,* and the set of all contexts are denoted by $\mathrm{Cont}(\mathcal{L}_\lambda)$. $(\Xi : \Gamma) \in \mathrm{Cont}(\mathcal{L}_\lambda)$ denotes a function f with domain Ξ and range Γ. If $f = (\Xi : \Gamma)$ is a context, and $x \in \Xi$ then $(x : \gamma)$ denotes $f(x) = \gamma$.

For a typed lambda language $\mathcal{L}_\lambda$ the sets of variables, expressions, contexts etc. defined and denoted by the same manner as for a lambda languages.

Definition 5. Let $\mathcal{L}_\lambda$ be a typed lambda language. By induction on the height of the construction tree of the expressions, relation

$$(\Xi : \Gamma) \vdash M : \varphi$$

will be defined as follows for every context $(\Xi : \Gamma) \in \mathrm{Cont}(\mathcal{L}_\lambda)$, expression $M \in \mathrm{Exp}(\mathcal{L}_\lambda)$ and type φ. $\vdash$ is called *typeability relation.*

(1) Let $|\mathrm{Tree}(M)| = 1$.
  (a) If $c \in \mathrm{Const}(\mathcal{L}_\lambda)$ and $(\Xi : \Gamma)$ is a context, then $(\Xi : \Gamma) \vdash c : \varphi$, if $\varphi = \mathrm{CnstTp}(\mathcal{L}_\lambda)(c)$.
  (b) If $x \in \mathrm{Var}(\mathcal{L}_\lambda)$ and $(\Xi : \Gamma)$ is a context, then $(\Xi : \Gamma) \vdash x : \varphi$, if $(x : \varphi) \in (\Xi : \Gamma)$.
(2) Let us suppose that $n > 1$ and for every $(\Upsilon : \Delta)$ context, type ψ and expression N with $|\mathrm{Tree}(N)| < n$, the relation $(\Upsilon : \Delta) \vdash N : \psi$ is defined. Let $(\Xi : \Gamma)$ be a context, φ a type and M an expression such that $|\mathrm{Tree}(M)| = n$.
  (a) Let $M = P(Q)$. Then $(\Xi : \Gamma) \vdash M : \varphi$, if $(\Xi : \Gamma) \vdash Q : \beta$ and $(\Xi : \Gamma) \vdash P : \alpha(\beta)$ and $\varphi = \alpha$.
  (b) Let $M = (\lambda x)P$. Then $(\Xi : \Gamma) \vdash M : \varphi$, if $(\Upsilon : \Delta) \vdash P : \alpha$, $\varphi = \alpha(\beta)$ and $(\Xi : \Gamma) = (\Upsilon : \Delta) \setminus \{(x : \beta)\}$.[11],

For some example, see [Troe, p. 10]

[11] Cf. Sorensen–Urzyczyn, p. 41, def. 3.1.1., Ibid: footnote 10.

2.2. **Montague semantics.**

Definition 6. Let $M \neq \emptyset$. By induction on $|\mathrm{Tree}(\varphi)|$, the domain set $D_M(\varphi)$ of the type $\varphi \in \mathcal{L}_{\mathrm{Typ}}$ is defined as follows.

(1) $D_M(o) = \{\mathsf{T}, \mathsf{F}\}, \qquad D_M(\iota) = M$
(2) If $D_M(\alpha)$ and $D_M(\beta)$ is defined earlier, then

$$D_M(\alpha(\beta)) = {}^{D_M(\beta)}D_M(\alpha)$$

where ${}^{D_M(\beta)}D_M(\alpha)$ is the set $\{f : D_M(\beta) \to D_M(\alpha)\}$.

If M is fixed, then $D(\varphi)$ is written instead.

Definition 7. If $M \neq \emptyset$, $\mathcal{L}_\lambda$ is a lambda language and $(\Xi : \Gamma)$ is a context, then a function $a : \mathrm{Var}(\mathcal{L}_\lambda) \to \cup_{\varphi \in \mathcal{L}_{\mathrm{Typ}}} D_M(\varphi)$ is an *assignation* of the variables. The assignation a is an *assignation of the type* $(\Xi : \Gamma)$, if for every $x \in \Xi$, $a(x) \in D_M(\alpha)$ whenever $(x : \alpha) \in (\Xi : \Gamma)$.

Definition 8. Let $\mathcal{L}_\lambda$ be a typed lambda language, $M \neq \emptyset$. The tuple $\mathfrak{M} = \langle M, \mathrm{Ip}^{\mathfrak{M}} \rangle$ is a *model* over the language $\mathcal{L}_\lambda$, if $\mathrm{Ip}^{\mathfrak{M}} : C \to \cup_{\varphi \in \mathcal{L}_{\mathrm{Typ}}} D(\varphi)$ such that $\mathrm{Ip}^{\mathfrak{M}}(c) \in D(\mathrm{CnstTp}(\mathcal{L}_\lambda)(c))$.

Definition 9. Let $\mathcal{L}_\lambda$ be a typed lambda language, $\mathfrak{M} = \langle M, \mathrm{Ip}^{\mathfrak{M}} \rangle$ a model over the language $\mathcal{L}_\lambda$, $(\Xi : \Gamma)$ a context and a an assignation of the type $(\Xi : \Gamma)$. Suppose, for $N \in \mathrm{Exp}(\mathcal{L}_\lambda)$ there is a type φ such that $(\Xi : \Gamma) \vdash N : \varphi$. By induction on $|\mathrm{Tree}(N)|$ the semantic value $[\![N]\!]^{\mathfrak{M}}_a$ in context $(\Xi : \Gamma)$ is defined as follows.

(1) If $N = c \in \mathrm{Const}(\mathcal{L}_\lambda)$, then

$$[\![c]\!]^{\mathfrak{M}}_a = \mathrm{Ip}^{\mathfrak{M}}(c).$$

(2) If $N = x \in \mathrm{Var}(\mathcal{L}_\lambda)$, then

$$[\![x]\!]^{\mathfrak{M}}_a = a(x).$$

(3) Let $N = P(Q)$, then

$$[\![P(Q)]\!]^{\mathfrak{M}}_a = [\![P]\!]^{\mathfrak{M}}_a([\![Q]\!]^{\mathfrak{M}}_a).$$

(4) Let $N = (\lambda x)P$ and
let the assignation $a[x \to \xi]$ be the following

$$a[x \to \xi](y) = a(y) \text{ for every variable } y \neq x, \text{ and } a(x) = \xi.$$

Then

$$[\![(\lambda x)P]\!]^{\mathfrak{M}}_a : D(\alpha) \to D(\beta)\ ; \xi \mapsto [\![P]\!]^{\mathfrak{M}}_{a[x \to \xi]}$$

where $(\Xi : \Gamma) \vdash x : \alpha$ and $(\Xi : \Gamma) \vdash P : \beta$.

Note that, if N is not typeable in a context $(\Xi : \Gamma)$, i.e. there is no type φ such that

$$(\Xi : \Gamma) \vdash N : \varphi$$

then N has no semantic value in an assignation of the type of the context. For example, let the type of the constant c be $o(\iota)$ and the context $(\Xi : \Gamma) = \{(x : o)\}$. Then the expression $c(x)$ is not typeable from the context $\{(x : o)\}$, since the argument of c must be an expression of the type ι. However, $c(x)$ is a well-defined expression, it has no semantic value in the context $\{(x : o)\}$.

2.3. **Logical and epsilon extensions.** The logical operators will be defined as constants of certain types. If $\mathcal{L}_\lambda$ is a typed lambda language, then it could be extended by the following constants.

(1) $\neg : o(o)$ $\quad \mathrm{Ip}^{\mathfrak{M}}(\neg) : \mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}$ in a model $\mathfrak{M}$,
(2) $\vee : o(o(o))$ $\quad \mathrm{Ip}^{\mathfrak{M}}(\vee) : (\mathsf{F}, \mathsf{F}) \mapsto \mathsf{F},$ and T otherwise in a model $\mathfrak{M}$,
(3) $\forall : o(o(\iota))$ $\quad \mathrm{Ip}^{\mathfrak{M}}(\forall) : \{\mathsf{T}, \mathsf{F}\}^M \to \{\mathsf{T}, \mathsf{F}\},\ (M \to \{\mathsf{T}, \mathsf{F}\}; \xi \mapsto \mathsf{T}) \mapsto \mathsf{T},$ and F otherwise in a model $\mathfrak{M}$,
(4) $\varepsilon : \iota(o(\iota))$ $\quad \mathrm{Ip}^{\mathfrak{M}}(\varepsilon) : \{\mathsf{T}, \mathsf{F}\}^M \to M : f \mapsto g(\{\xi \in M \mid f(\xi) = \mathsf{T}\})$,
where g is a fixed *choice function* $\mathcal{P}(M) \to M$ such that $g(S) \in S$, if $S \neq \emptyset$ and $g(S) \in M$, if $S = \emptyset$, in a model $\mathfrak{M}$.

In the following two specific extension will be mentioned, the *plain extension*

$$\mathcal{L}_\lambda^\forall \text{ with } \mathrm{Const}(\mathcal{L}_\lambda^\forall) = \mathrm{Const}(\mathcal{L}_\lambda) \cup \{\neg, \vee, \forall\}$$

and the *epsilon extension*

$$\mathcal{L}_\lambda^{\forall\varepsilon} \text{ with } \mathrm{Const}(\mathcal{L}_\lambda^{\forall\varepsilon}) = \mathrm{Const}(\mathcal{L}_\lambda) \cup \{\neg, \vee, \forall, \varepsilon\}.$$

If $\mathfrak{M}$ is a model of $\mathcal{L}_\lambda^\forall$, then $(\mathfrak{M}, g)$ will denote the (expanded) model of the $\mathcal{L}_\lambda^{\forall\varepsilon}$ extension with a choice function g described above.[12]
Some further (classical) notations will be used:

$$P \to Q = \vee([\neg(P)](Q)), \qquad P\&Q = \neg(\vee([\neg(P)](\neg(Q)))), \qquad (\forall x)P = \forall((\lambda x)P)$$

$$(\varepsilon x)P = \varepsilon((\lambda x)P).$$

For further purposes the language $\mathcal{L}_\lambda^{\forall\varepsilon=}$ using *identity* of individuals is also introduced and the meaning of $=$ is defined as

(5) $=: (o(\iota))(\iota)$ $\quad \mathrm{Ip}^{\mathfrak{M}}(=) : M^2 \to \{\mathsf{T}, \mathsf{F}\}, (x, y) \mapsto \mathsf{T},$ if $x = y$ and F otherwise in a model $\mathfrak{M}$.

2.4. **Examples.**

Proposition 1. Let x be a variable and $(\mathfrak{M}, g)$ be model over the language $\mathcal{L}_\lambda^{\forall\varepsilon=}$. Then

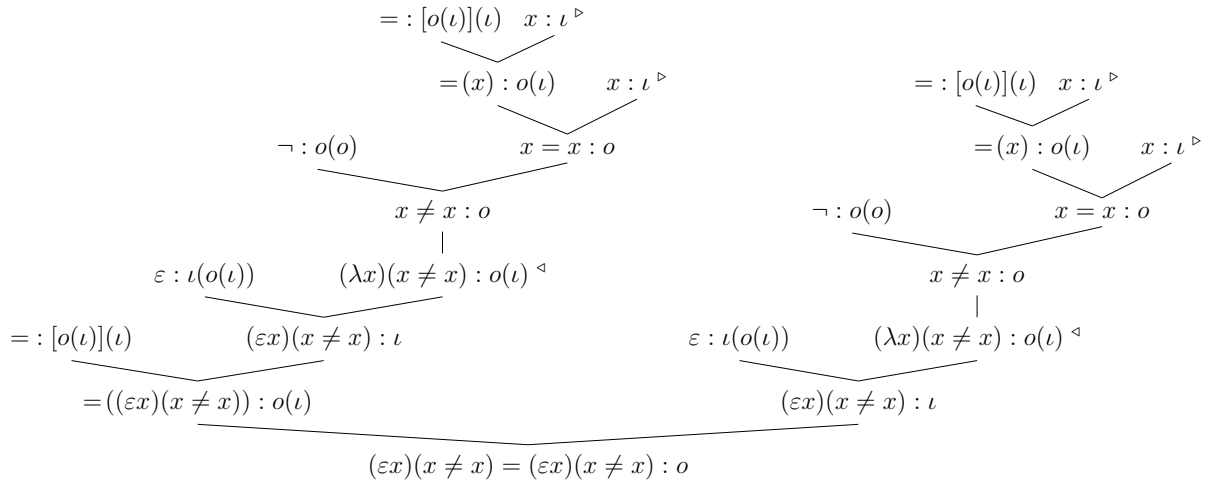
(1) $\vdash (\forall x)(x = x) : o$
(2) $\vdash (\varepsilon x)(x \neq x) = (\varepsilon x)(x \neq x) : o$
(3) $[[(\forall x)(x = x)]]^{(\mathfrak{M},g)} = [[(\varepsilon x)(x \neq x) = (\varepsilon x)(x \neq x)]]^{(\mathfrak{M},g)} = \mathsf{T}$

Proof. (1)

$$\dfrac{\forall : o(o(\iota)) \qquad \dfrac{\dfrac{=: [o(\iota)](\iota) \quad x : \iota^{\,\triangleright}}{=(x) : o(\iota)} \qquad x : \iota^{\,\triangleright}}{x = x : o}}{\forall((\lambda x)(x = x)) : o}\ (\lambda x)(x = x) : o(\iota)^{\,\triangleleft}$$

[12]Actually, epsilon-terms are special kind of Skolem functions; it is pointed out in [Monk, The Hilbert ε-operator, p. 481] and in [Mints, Sec. 2.: Quantifier-Free Extensions of Formulas and ε-Theorems)].

Here $(= (x))(x)$ is denoted by $x = x$. The proof tree above shows that the expression $\forall((\lambda x)(x = x))$, which is the same as $(\forall x)(x = x)$, is typeabe by the type o. The labels $^{\triangleright}$ on the left sides of the leaves mark the places which are called the "dischargeable premises" in proof theory. Their role is abandoned at the node labeled by $^{\triangleleft}$. According to the part (2b) of Definition 5, both the $x : \iota$-s are discharged by the node $(\lambda x)([= (x)](x)) : o(\iota)$, i.e. $(x : \iota)$ can be canceled from the context, which is now an empty set. Note that, the use of the labels $^{\triangleleft}$ and $^{\triangleright}$ are completely unnecessary, since the role of the variable x is exactly that of the triangles. The variable x in the leaves marks the "dischargeable premises" and the symbol (λx) marks the node discharging the premises labeled by the free variable x, and then x becomes a bound variable.
(2)



Here, according to definition, $\varepsilon((\lambda x)(x \neq x))$ is denoted by $(\varepsilon x)(x \neq x)$ and $\neg(x = x)$ is denoted by $x \neq x$. The above proof tree proves that $(\varepsilon x)(x \neq x) = (\varepsilon x)(x \neq x)$ is typeable by o.
(3)

$$\begin{aligned}
[[(\forall x)(x = x)]]_a^{(\mathfrak{M},g)} &= [[\forall((\lambda x)(x = x)]]_a^{(\mathfrak{M},g)} \\
&= [[\forall]]_a^{(\mathfrak{M},g)}([[(\lambda x)(x = x)]]^{(\mathfrak{M},g)})_a \\
&= [[\forall]]_a^{(\mathfrak{M},g)}(\xi \mapsto [[= (x)(x)]]_{a[x\to\xi]}^{(\mathfrak{M},g)}) \\
&= [[\forall]]_a^{(\mathfrak{M},g)}(\xi \mapsto [[= (x)]]_{a[x\to\xi]}^{(\mathfrak{M},g)}(\xi)) \\
&= [[\forall]]_a^{(\mathfrak{M},g)}(\xi \mapsto [[=]]_{a[x\to\xi]}^{(\mathfrak{M},g)}(\xi)(\xi)) \\
&= [[\forall]]_a^{(\mathfrak{M},g)}(\xi \mapsto \mathsf{T}) \\
&= \mathsf{T}
\end{aligned}$$

The second expression's semantic value is trivial:

$$\begin{aligned}
[[(\varepsilon x)(x \neq x) = (\varepsilon x)(x \neq x)]]_a^{(\mathfrak{M},g)} &= [[=]]_a^{(\mathfrak{M},g)}([[(\varepsilon x)(x \neq x)]]_a^{(\mathfrak{M},g)})([[(\varepsilon x)(x \neq x)]]_a^{(\mathfrak{M},g)}) \\
&= \mathsf{T}
\end{aligned}$$

however, it is worth to determine the epsilon term's value:

$$\begin{aligned}
[[(\varepsilon x)(x \neq x)]]_a^{(\mathfrak{M},g)} &= [[\varepsilon((\lambda x)(x \neq x))]]_a^{(\mathfrak{M},g)} \\
&= [[\varepsilon]]_a^{(\mathfrak{M},g)}([[(\lambda x)(x \neq x)]]_a^{(\mathfrak{M},g)}) \\
&= [[\varepsilon]]_a^{(\mathfrak{M},g)}(\xi \mapsto [[x \neq x]]_{a[x\to\xi]}^{(\mathfrak{M},g)}) \\
&= g(\{\xi \in M \mid [[x \neq x]]_{a[x\to\xi]}^{(\mathfrak{M},g)} = \mathsf{T}\}) = g(\emptyset) \\
&= g(\{\xi \in M \mid [[\neg]]_{a[x\to\xi]}^{(\mathfrak{M},g)}([[=]]_{a[x\to\xi]}^{(\mathfrak{M},g)}(\xi)(\xi)) = \mathsf{T}\}) = g(\emptyset)
\end{aligned}$$

□

2.5. **Epsilon-invariant expressions.**

Definition 10. Let $N \in \mathrm{Exp}(\mathcal{L}_\lambda^{\forall\varepsilon})$ be such that for a context $(\Xi : \Gamma)$ the relation $(\Xi : \Gamma) \vdash N : \varphi$ holds for a type φ and let $\mathfrak{M}$ be a $\mathcal{L}_\lambda^{\forall}$ model. N is said to be *epsilon-invariant over the model* $\mathfrak{M}$, if for every assignation a of the type $(\Xi : \Gamma)$ and choice functions $g_1, g_2 : \mathcal{P}(M) \to M$ it holds that

$$[\![N]\!]_a^{(\mathfrak{M},g_1)} = [\![N]\!]_a^{(\mathfrak{M},g_2)}.$$

The notion above is a symbolic formulation of the intuitive term "epsilon-independent". In FOL this concept was applied to show that "epsilon-independent" sentences can be reformulated into an epsilon-free one, provided the sentence is independent over *every* model.[13]

3. Epsilon and application

Theorem 1. Let $P, Q \in \mathrm{Exp}(\mathcal{L}_\lambda^{\forall\varepsilon})$, $\mathfrak{M}$ be a model of $\mathcal{L}_\lambda^{\forall}$, $(\Xi : \Gamma)$ a context, $(\Xi : \Gamma) \vdash P : o$, $(\Xi : \Gamma) \vdash Q : o$ and $x \in \mathrm{Var}(\mathcal{L}_\lambda^{\forall\varepsilon})$, furthermore, let $[(\lambda x)P]((\varepsilon x)Q)$, P and Q be epsilon-invariant over the model $\mathfrak{M}$. Then for every assignation a of the type $(\Xi : \Gamma)$ and choice function $g : \mathcal{P}(M) \to M$:

$$[\![[(\lambda x)P]((\varepsilon x)Q)]\!]_a^{(\mathfrak{M},g)} = [\![((\forall x)(\neg Q)\&(\forall x)P) \vee (((\exists x)Q)\&(\forall x)(Q \to P))]\!]_a^{(\mathfrak{M},g)}.$$

Proof. (1) Let the right hand side be T. First case: $[\![((\forall x)(\neg Q)\&(\forall x)P)]\!]_a^{(\mathfrak{M},g)} = \mathsf{T}$. Then $[\![(\forall x)P)]\!]_a^{(\mathfrak{M},g)} = \mathsf{T}$ holds and let $m = [\![(\varepsilon x)Q]\!]_a^{(\mathfrak{M},g)} \in M$. Hence, by definition

$$\mathsf{T} = [\![(\forall x)P)]\!]_a^{(\mathfrak{M},g)} = [\![\forall((\lambda x)P)]\!]_a^{(\mathfrak{M},g)}$$

that is

$$[\![(\lambda x)P]\!]_a^{(\mathfrak{M},g)} = \left(\xi \mapsto [\![P]\!]_{a[x\to\xi]}^{(\mathfrak{M},g)}\right) \equiv \mathsf{T}.$$

Hence

$$[\![[(\lambda x)P]((\varepsilon x)Q)]\!]_a^{(\mathfrak{M},g)} = [\![[(\lambda x)P]]\!]_a^{(\mathfrak{M},g)}(m) = [\![P]\!]_{a[x\to m]}^{(\mathfrak{M},g)} = \mathsf{T}.$$

Second case: $[\![(((\exists x)Q)\&(\forall x)(Q \to P))]\!]_a^{(\mathfrak{M},g)} = \mathsf{T}$. Then

$$[\![(\lambda x)\neg Q]\!]_a^{(\mathfrak{M},g)} = \left(\xi \mapsto [\![\neg Q]\!]_{a[x\to\xi]}^{(\mathfrak{M},g)}\right) \not\equiv \mathsf{T}$$

hence for a $\xi \in M$ $[\![Q]\!]_{a[x\to\xi]}^{(\mathfrak{M},g)} = \mathsf{T}$. Therefore, if $[\![\varepsilon((\lambda x)Q)]\!]_a^{(\mathfrak{M},g)} = m$ then $[\![(\lambda x)Q]\!]_a^{(\mathfrak{M},g)}(m) = \mathsf{T}$. But from $[\![(\forall x)(Q \to P))]\!]_a^{(\mathfrak{M},g)} = \mathsf{T}$ it follows that $[\![P]\!]_{a[x\to m]}^{(\mathfrak{M},g)} = \mathsf{T}$, since $[\![Q]\!]_{a[x\to m]}^{(\mathfrak{M},g)} = \mathsf{T}$. Hence, $[\![(\lambda x)P]\!]_a^{(\mathfrak{M},g)}(m) = [\![[(\lambda x)P]((\varepsilon x)Q)]\!]_a^{(\mathfrak{M},g)} = \mathsf{T}$.
(2) Suppose the left hand side is T. First case: let $[\![((\forall x)(\neg Q)]\!]_a^{(\mathfrak{M},g)} = \mathsf{T}$, $m \in M$ arbitrary and g' is

[13] See [Blas].

the choice function such that $g'(\emptyset) = m$. Hence, by the epsilon-invariance of P and $[(\lambda x)P]((\varepsilon x)Q)$ it follows that

$$\mathsf{T} = [\![[(\lambda x)P]((\varepsilon x)Q)]\!]_a^{(\mathfrak{M},g)} = [\![[(\lambda x)P]((\varepsilon x)Q)]\!]_a^{(\mathfrak{M},g')} = [\![P]\!]_{a[x\to m]}^{(\mathfrak{M},g')} = [\![P]\!]_{a[x\to m]}^{(\mathfrak{M},g)}$$

therefore $[\![(\forall x)P]\!]_a^{(\mathfrak{M},g)} = \mathsf{T}$. Second case: let $[\![((\exists x)Q]\!]_a^{(\mathfrak{M},g)} = \mathsf{T}$, $m \in M$ arbitrary such that $[\![Q]\!]_{a[x\to m]}^{(\mathfrak{M},g)} = \mathsf{T}$ and g' is the choice function such that $g'(\{\xi \in M \mid [\![Q]\!]_{a[x\to\xi]}^{(\mathfrak{M},g)} = \mathsf{T}\}) = m$. Then by the epsilon-invariance of P, Q and $[(\lambda x)P]((\varepsilon x)Q)$ it follows that

$$\mathsf{T} = [\![[(\lambda x)P]((\varepsilon x)Q)]\!]_a^{(\mathfrak{M},g)} = [\![[(\lambda x)P]((\varepsilon x)Q)]\!]_a^{(\mathfrak{M},g')} = [\![P]\!]_{a[x\to m]}^{(\mathfrak{M},g')} = [\![P]\!]_{a[x\to m]}^{(\mathfrak{M},g)}$$

for every m such that $[\![Q]\!]_{a[x\to m]}^{(\mathfrak{M},g)} = \mathsf{T}$. Hence, $[\![(\forall x)(Q \to P)]\!]_a^{(\mathfrak{M},g)} = \mathsf{T}$

□

4. Morning Star and King of France Tests

The concluding facts can be stated in two claims:

(1) In the formal language $\mathcal{L}_\lambda^{\forall\varepsilon}$ (which is supposed to model the behaviour of descriptions) the (closed) term $(\varepsilon x)Q$ has *referential meaning* in the sense that a fixed model $(\mathfrak{M}, g)$ points out an individual $[\![(\varepsilon x)Q]\!]^{(\mathfrak{M},g)} \in M$ for $(\varepsilon x)Q$ as semantic value.
(2) In some cases, when $(\varepsilon x)Q$ is a part of a compound sentence $[(\lambda x)P]((\varepsilon x)Q)$, with all its components being epsilon-invariant, the $(\varepsilon x)Q$ has a *contextual meaning too,* such that the sentence $[(\lambda x)P]((\varepsilon x)Q)$ has an equivalent epsilon-free reformulation using quantified expressions from the plain language $\mathcal{L}_\lambda^{\forall}$.

We do not intend to set up a theory which is receiving fewer results than Russell's Theory of Descriptions. A new theory must serve at least as many solutions as far as Russell's proposal was able to solve. An appropriate indicator is to look at the two problems that the Theory of Descriptions solved and examine what the new model results in this field. The first one is the problem of Hesperus and Phosphorus (below, it will be called Morning Star Test), the second one is the problem of the empty names (the King of France Test).

4.1. **Morning Star Test.** In 1905, Russell gave a FOL based solution of the so called Frege Puzzle by RTD, understandably, without mentioning the intensional tools of possible world semantics, which is a much later development. Here, I would like to show briefly that even the exposition of the puzzle is so widely criticized, that the RTD result of the test is rather irrelevant to us.

> "Gottlob thinks that the Morning Star is illuminated by the Sun."
> "The Evening Star is the Morning Star."
> —
> "Gottlob thinks that the Evening Star is illuminated by the Sun". (Cf. [Freg].)

First of all, I would like to point out that several scholars are committed to the standpoint that the names such as "the Morning Star" or "the Evening Star" are understood tacitly as definite descriptions. For Russell, these names abbreviate descriptions, hence they are denoting phrases too.[14] The problem is that, according to Leiniz's Rule, since the Evening Star is the Morning Star, the two phrases are interchangeable. However, the above inference does not seem to be valid, since it is possible that Gottlob thinks that the Morning Star is illuminated by the Sun, but he does not necessarily know this fact about the Evening Star, even if in reality the two planets are the same, which is the case. Russell's solution was that the phrases "the Morning Star" and "the Evening Star" are not proper names, they only have contextual meanings, hence they are not interchangeable due to formal reasons.[15]

[14][Dumm, p. 97].
[15]See [Russ] and [Whit].

In the epsilon language $\mathcal{L}_{\lambda}^{\forall\varepsilon}$, the definite descriptions are proper names, they are manifested as epsilon terms on the object language level, hence the epsilon modelling fails the Morning Star Test, it does not explain the puzzle. Fortunately, hitherto, the Frege Puzzle and the semantic status of the expressions like "the Morning star" are not completely solved. If the phrase "the Morning star" is a rigid designator, what is Kripke's proposal, then the Puzzle is solved. Here, temporarily, not having modal context, 'rigid' means that the model points out a single individual immediately, and does not select first a set, then a member of it, by a choice function.[16] Then the puzzle only says that, if planet Venus is illuminated by the Sun, then planet Venus is illuminated by the Sun. According to Kripke's solution the problematic one is the sentence "The Evening Star is the Morning Star". It is a necessary truth, but it may be epistemologically problematic.[17] For the epsilon model, the solution is the same. According to Monk the closed epsilon terms are constants, therefore they are rigid designators in accordance with the Kripke doctrine. However, as Fitting pointed out, an epsilon term, being description-like, can neither be a constant, nor a variable. It is a complex flexible designator.[18] Here, if "the Morning star" is a complex demonstrative (selected by a descriptive term in the actual world), then it is a rigid designator.[19] Clearly, now, I do not have to deal with the modal context of epsilon terms, knowing that the highly applicable tool of demonstratives might make the modal approach much more complex, and might not add essentially more to the above consideration.

4.2. **The King of France Test.** Consider the following two sentences

"The present King of France is bald."

"The present King of France is not bald."

In order to determine the truth value of the first one, let us imagine the set of all bald people. Since the present King of France is not in this set, the first sentence is false. But, the same reasoning leads to the fact that the second sentence is false too. Which is a contradiction. Hence, the phrase "the present King of France" is not a proper name, it cannot have a meaning in isolation, rather it only has a contextual meaning and the sentences containing such phrases are quantified formulas. This is Russell's solution.

In the epsilon calculus the semantic values of the epsilon terms are defined in any cases. The two sentences above are unproblematic, having the phrase "the present King of France" an existing individual as reference. And it is either bald or not bald. According to Theorem 1 of the present paper, the sentences *may* possess contextual meaning too, where the truth value is also well-defined. Of course, the reference of "the present King of France" in the epsilon calculus is not the present King of France. Approaching the situation on a more formal level, let us consider the symbolic sentence

$$(\varepsilon x)(x \neq x) = (\varepsilon x)(x \neq x)$$

This is a sentence containing terms which are ill-defined as descriptions: $x \neq x$ is an empty predicate. However, the semantic value of $(\varepsilon x)(x \neq x)$, in a given model, is well-defined. Moreover, $(\varepsilon x)(x \neq x) = (\varepsilon x)(x \neq x)$ is an epsilon invariant sentence, since, it is true in any given epsilon semantics. And indeed, there are epsilon semantics (for example the Bourbaki group's formal systems), where $(\varepsilon x)(x \neq x) = (\varepsilon x)(x \neq x)$ is syntactically identical to the sentence $(\forall x)(x = x)$. $(\varepsilon x)(x \neq x) = (\varepsilon x)(x \neq x)$ is an epsilon-invariant sentence, which has contextual meaning too: it is equivalent to the fact that every individual is identical to itself.

The situation is very similar to the problem of the interesting-looking man holding a martini. In this case, the "the present King of France" is rather a person who is, in fact, bald, but not the present King of

[16]Of course, it is a rough simplification. Picking an individual means direct reference, rigid means the term has the same semantic value along the possible worlds. What is more, the notion of 'rigid' above is understandable, but mathematically vague.

[17]See [Krip, p. 102]. The whole story can be found in [Zvol].

[18]See [Fitt].

[19]See [Kapl].

France, and $(\varepsilon x)(x \neq x)$ is an existing individual, which is identical to itself, but of course, it does not hold that it is not the same as itself.

ACKNOWLEDGEMENT

I would like to thank Endre Latabár, András Simon and the unknown reviewer for their helpful comments concerning specific parts of the paper or the whole one.

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DEPARTMENT OF ALGEBRA, BUDAPEST UNIVERSITY OF TECHNOLOGY AND ECONOMICS, 1111 EGRY JÓZSEF U. 1. BUILDING H, 5^{th} FLOOR, BUDAPEST, HUNGARY

E-mail address: mozow@math.bme.hu

URL: http://www.math.bme.hu/~mozow/